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MULTI-PHASE-MISSION RELIABILITY

OF MAINTAINED STANDBY SYSTEMS

by

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In a phased mission the functional organization of the system changes at selected times which mark the boundaries of the phases of the mission. Existing methods for analysis of phased missions are modified and extended to permit determination of the reliability of systems which are maintained during a standby period, called the operational readiness phase, during which the system functions solely to maintain its readiness for a later period of active operations. This mode of operation is frequently exhibited

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by strategic weapon systems and safety devices. These results are then extended to systems which perform complex multi-objective missions to permit assessment of system performance at levels intermediate between total failure and total success.

1. INTRODUCTION

The concept of a phased mission was introduced in the early papers of Rubin [1964] and Weisburg and Schmidt [1966], which were motivated by the need for reliability and crew-safety predictions in the manned space programs. In a recent paper, Esary and Ziehms [1975] provide a mathematical description of the phased-mission problem and show that the performance of a system of non-maintained components with a multi-phase mission can be analyzed by considering an equivalent system with a single-phase mission. The transformation from a multi-phase mission to a single-phase mission makes possible the application of standard structural reliability techniques.

The elements of the phased-mission problem to be considered here are described in the following situation.

> A *system* consists of several *components* which perform independently of each other. Each component is said to be *functioning* (up) if it performing satisfactorily; otherwise it is said to be *failed* (down). The system performs a *mission* which can be divided into an active portion and a standby portion. The standby portion of the mission is called the *operational readiness (O R) phase*. The active portion is further divided into consecutive time periods or *phases* of known duration during which the functional organization of the system does not change. Implicit in the specification of the mission are the levels of performance which are satisfactory and the environment in which it will be undertaken. The functional organization for each phase can be represented as a *block diagram*

built of a subset of the system's components (or equivalently as a *fault tree*). The system is designed to accomplish a specific task during each phase, and the mission has one or more goals or *objectives* about which it is desired to make probability statements.

The phased-mission problem as described above introduces two concepts not previously considered in this context--the O R phase and the possibility of multiple mission objectives. The duration of the O R phase will generally be unknown since the timing of the event which causes its termination will be difficult to forecast. Some system components will usually be required to function during the O R phase in order to maintain the system's readiness. Because of the prolonged nature of this phase it is highly likely that failures will occur among these active components and possibly among those that are dormant. Thus it is reasonable to expect that some means of monitoring components and correcting failures will be provided. Previous work on the phased-mission problem has been limited to non-maintained systems. When systems are not maintained, the usual measure of the performance of a system or component is its reliability--that is, the probability that it will perform satisfactorily in the prescribed environment for the period intended. If a system's components are subject to maintenance, the mission reliability cannot be expressed as a function of component reliabilities alone. Usually, the incorporation of maintenance actions in a model of system performance adds significantly to the complexity of the problem. When maintenance is performed only during the O R phase, however, the phased-mission problem retains much of its simplicity.

The notion of multiple objectives serves primarily as a means of recognizing more than two levels of system performance. It is a

general concept which can be tailored to the individual application, and it facilitates the formulation of more incisive reliability statements than those afforded by the usual binary (success/failure) measure of system performance. This concept is used in a highly-structured context in Chapter 3.

Examples of situations which fit the description of the phased-mission problem are numerous. Safety systems, in particular, provide prime examples. Even as simple a system as the local fire station contains all of the ingredients described above. Many military systems, especially strategic weapon systems, are designed to remain in readiness until activated in response to a threat to the national security. Indeed, many of these systems are never activated throughout their entire lifespans.

Much of the work in this paper was motivated by the author's conception of the Navy's Fleet Ballistic Missile (FBM) weapon system. This system consists of a nuclear-powered submarine and its associated subsystems and sixteen ballistic missiles, each containing many subsystems of its own. The O R phase for any one of the systems commences when it relieves its predecessor in a patrol area and terminates when it is relieved in turn or when a command to launch a missile strike is issued. System components are monitored and maintained throughout the O R phase. Upon activation, the system proceeds through phases during which missiles are prepared for launch, the submarine is positioned, and the missiles ejected. As each missile is ejected, it becomes independent of the submarine-borne subsystems and itself progresses through several phases of flight. Each missile is assigned to a target (or more than one target if it has multiple warheads), and destruction of

that target is one of the mission objectives. Thus this system is one with sixteen or more objectives and several active phases following an extended O R phase. This situation is one in which it is natural to seek the expected value or probability distribution of the number of objectives attained (targets destroyed) during the mission. These notions are among those which are discussed in Chapter 3.

2. THE SINGLE-OBJECTIVE, PHASED-MISSION WITH AN O R PHASE

The situation considered in this chapter is that of a system which performs a single-objective mission with an O R phase and several active phases. A mathematical model is constructed in several stages which relates system performance to that of its components. The development uses the standard tools of structural reliability and extends and modifies the model of Esary and Ziehms [1975]. The various aspects of the extended phased-mission problem are illustrated in the following section by an example motivated by the Fleet Ballistic Missile system.

2.1 THE SLBM SYSTEM EXAMPLE

The following example introduces a hypothetical system which will serve both as motivation and illustration of the model development throughout this chapter and again in Chapter 3.

Example 2.1. A hypothetical submarine-launched ballistic missile system (SLBM) consists of the following components:

--the submarine (S) which provides propulsion, stability, power, and household services.

--the inertial navigation subsystem (N) which provides information on platform position and orientation.

--the communication subsystem (C) which provides the link between the submarine and its command center.

--the fire control subsystem (FC) which provides trajectory information to the missile guidance computer.

--the missile ejection subsystem (E) which launches the missile from the submarine while the latter is submerged.

--the missile guidance component (G) which computes and transmits to the rocket engines the control commands required to maintain the trajectory stored within its memory and triggers stage separation.

--two missile internal power sources (VP and VS).

--the first and second stage rocket engines (RF and RS).

--the first-stage igniters (IP and IS).

--the second-stage igniter (J).

--the missile warhead (W).

The operational characteristics of the system can be summarized as follows:

(a) During the operational readiness phase the submarine patrols its assigned area, maintaining current position information with the inertial navigation subsystem. Should the inertial component fail, then position information can be obtained periodically from a navigation satellite, providing the data necessary for calibration after repairs are completed. The communication subsystem is used continually during the phase for routine ship-shore message traffic. The fire control subsystem is exercised periodically during the O R phase to monitor its status. Similarly, the performance of the missile power sources and guidance component is checked through routine tests. All components which are monitored can be repaired or replaced if found to be failed during the O R phase. Other failures go undetected. In order for the system to be ready to commence active operations, it must have submarine services and current navigation information available, and it must be able to receive the launch command via the communication subsystem.

(b) When a launch command is received, all maintenance actions

cease, and launch preparations commence. The fire control subsystem transmits trajectory data to the missile guidance component, and the submarine is positioned for launch.

(c) During the launch phase the submarine is held stable while the missile is ejected, severing its link with the platform and causing it to switch to internal power. The power sources, although activated, are not required to supply power during this phase.

(d) The first-stage engine ignites as the missile breaks through the surface of the water and boosts the missile along its trajectory. The port igniter can be powered by only the port power source and the starboard igniter by only the starboard power source, but one igniter is sufficient to fire the engine. The guidance component, which can take power from either source, must function throughout the phase.

(e) The second-stage igniter, second-stage engine, guidance component, and at least one power source must function during the flight phase.

(f) Shutdown of the second-stage engine marks the beginning of the terminal phase during which the warhead follows a ballistic trajectory to the target. □

2.2 THE EXTENDED PHASED-MISSION MODEL

The mission consists of an O R phase followed by m active phases. The O R phase commences at time $t=0$ and continues until time t_0 when active operations begin. For $j=1,\ldots,m$, the duration of active phase j is assumed to be d_j. Recognizing that t_0 is unknown, let

$$t_j = \sum_{i=1}^{j} d_i + t_0,$$

$j=1,\ldots,m$. Thus t_j is the time at which phase j ends and (except when

$j = m$) the next phase begins.

The system has n components (or subsystems) $C_1, \ldots, C_n$, which function independently of each other. Assigned to each component C_k, $k=1,\ldots,n$, is a Bernoulli *performance state indicator variable* $X_k(t)$ defined for all $t \geq 0$ by

$$X_k(t) = \begin{cases} 1 \text{ if } C_k \text{ is functioning at time } t, \\ 0 \text{ otherwise.} \end{cases}$$

The stochastic process $\{X_k(t), t \geq 0\}$ is called the *performance process* of component C_k, and the multivariate stochastic process

$$\{\underline{X}(t), t \geq 0\} = \{[X_1(t), \ldots, X_n(t)], t \geq 0\}$$

is the *joint component performance process* of the system.

As in Example 2.1, it will generally be the case that some portion of the system's components can be repaired or replaced upon failure during the O R phase; however it is assumed that no system maintenance is performed after time t_0. Then the performance process of each repairable component C_k has the properties:

$$\begin{aligned} &X_k(t) = 0 \Leftrightarrow X_k(s) = 0, \text{ for all } s > t \geq t_0 \\ (2.2.1) \qquad &X_k(t) = 1 \Leftrightarrow X_k(s) = 1, \text{ for all } s \geq t_0 \text{ such that } s \leq t. \end{aligned}$$

The performance processes of the remaining components satisfy these relations when t_0 is replaced by 0. Thus a sample path of the performance process for a non-repairable component is non-increasing and continuous from the right, and that for a repairable component is also continuous from the right and is non-increasing after time t_0 as shown in Figure 2.1.

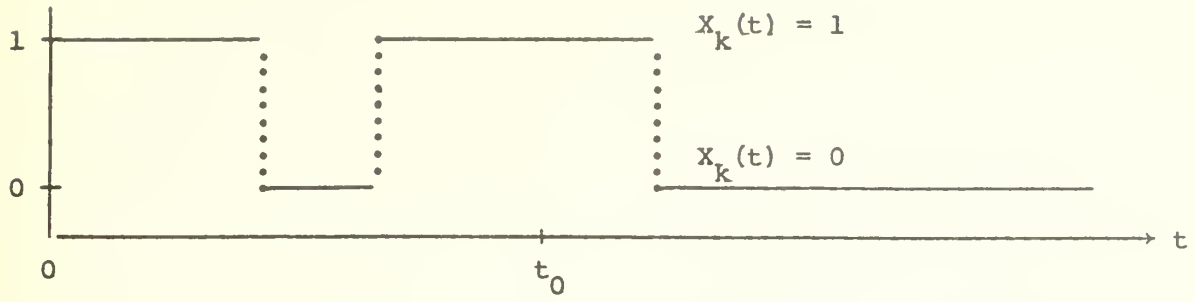


Figure 2.1. Performance process sample path of repairable component C_k.

The joint component performance process is a complete mathematical description of the performance of the system's components. It is useful to summarize the characteristics of a component's performance process in the form of probability statements. The reliability of a component is a statement about its performance *over a period of time*. Thus the reliability of component C_k during the period $[t, t + d]$ is defined as $P[X_k(s) = 1, t < s \leq t+d]$. There are also instances when it is desired to make statements about the performance of a component *at a point in time*. Hence the *availability* of component C_k *at time t* is defined by

$$\alpha_k(t) = P[X_k(t) = 1]$$

If the performance process is non-increasing over the interval $[t, t+d]$ then the reliability of component C_k over the period is equal to its availability at time $t+d$ by Relations 2.2.1.

The state of the system at any time is assumed to be completely determined by the states of its components. The system structure is the connecting link. In a phased mission this structure does not remain fixed throughout the mission, but is allowed to change from phase to phase. Thus, letting the O R phase be phase zero, there is a binary

structure function ϕ_j of the binary variables $x_1, \ldots, x_n$ for each phase j, j=0,...,m, defined by

$$\phi_j(x_1, \ldots, x_n) = \begin{cases} 1 \text{ if the system functions, and} \\ 0 \text{ otherwise.} \end{cases}$$

The composition $\phi[\underline{X}(t)]$ where ϕ is defined by

$$\phi[\underline{X}(t)] = \begin{cases} \phi_0[\underline{X}(t)], & 0 < t \le t_0 \\ \phi_1[\underline{X}(t)], & t_0 < t \le t_1 \\ \vdots & \\ \phi_m[\underline{X}(t)], & t_{m-1} < t \le t_m \end{cases}$$

is itself a Bernoulli random variable called the *system performance indicator variable*. The corresponding stochastic process $\{\phi[\underline{X}(t)], t>0\}$ is called the *system performance process*. Although the sample paths of $\{\phi[\underline{X}(t)], t > 0\}$ are not necessarily right continuous, the right continuity of the sample paths of the component performance processes leads to right continuity of the system performance process sample paths within each phase.

In order for the system to satisfactorily complete its mission, it must function throughout each active phase. The O R phase, however, is different, for it is merely a readiness period. It is not necessary that the system function throughout this phase. The single requirement is that the system be available when the O R phase ends. Thus the mission reliability is given by

$$p = P\{\phi_0[\underline{X}(t_0)] = 1, \phi_1[\underline{X}(s_1)] = 1, t_0 < s_1 \le t_1, \ldots, \phi_m[\underline{X}(s_m)] = 1, t_{m-1} < s_m \le t_m\} \quad (2.2.2)$$

The structure of a system is typically represented as a *block diagram* of its components (or equivalently as a *fault tree*). Structures which can be so depicted belong to a special class whose structure functions are said to be *coherent*. Birnbaum-Esary-Saunders [1961] defined a coherent structure function to be one for which

(2.2.3)
(a) $\phi(\underline{x}) \geq \phi(\underline{y})$ whenever $x_k \geq y_k$, $k=1,\ldots,n$,
(b) $\phi(\underline{0}) = \phi(0,0,\ldots,0) = 0$, and
(c) $\phi(\underline{1}) = \phi(1,1,\ldots,1) = 1$.

Since nearly all physical systems have a block-diagram representation, it is assumed that the structure function for each active phase is coherent.

The general model makes provision for the inclusion of a structure function for the O R phase. The system of Example 2.1 provides an illustration of circumstances in which a phase 0 structure function is appropriate, since active operations cannot commence unless certain components are available. The following example shows a case in which use of the function $\phi_0 = 1$ is appropriate.

<u>Example 2.2.</u> A system with a three-phase mission has two components, C_1 and C_2, both of which are dormant during the O R phase. During the first active phase the system functions only if both components function, but during the second phase the functioning of either component will allow the system to function. The structure functions for this mission are:

$$\text{for phase 0,} \quad \phi(x_1, x_2) = 1$$

$$\text{for phase 1,} \quad \phi(x_1, x_2) = x_1 x_2$$

$$\text{for phase 2,} \quad \phi(x_1, x_2) = x_1 \vee x_2$$

where the symbol $\vee$ is the arithmetic "or" operator defined by

$$x_1 \vee x_2 = \begin{cases} 1 \text{ if } x_1 = 1 \text{ or } x_2 = 1, \\ 0 \text{ if } x_1 = 0 \text{ and } x_2 = 0 \end{cases}$$

Computationally, $x_1 \vee x_2 = x_1 + x_2 - x_1x_2$. The corresponding block diagrams for the phases of this mission are shown in Figure 2.2. □

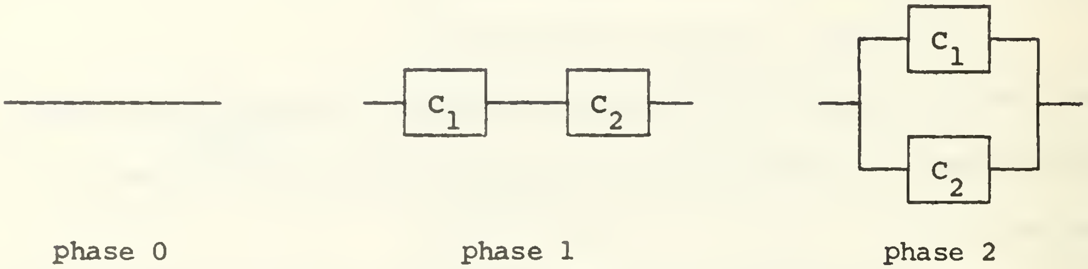


Figure 2.2. Block diagrams for the mission of Example 2.2.

In general, the O R phase structure function will be $\phi_0 = 1$ unless one or more of the components is actively required at time t_0, in which case ϕ_0 will be a coherent structure function. Thus it is assumed that ϕ_0 is at least *semi-coherent* (satisfies Relation 2.2.3a) in all cases.

Since the joint component performance process is non-increasing after time t_0, it follows from Relation 2.2.3a that the system performance process is also non-increasing within each phase. Thus the mission reliability as given by Equation 2.2.2 reduces to the less complex expression

$$p = P\{\phi_0[\underline{X}(t_0)] = 1,\ \phi_1[\underline{X}(t_1)] = 1, \ldots, \phi_m[\underline{X}(t_m)] = 1\}$$

or more simply

$$p = P\{\prod_{j=0}^{m} \phi_j[\underline{X}(t_j)] = 1\} = E\{\prod_{j=0}^{m} \phi_j[\underline{X}(t_j)]\} \tag{2.2.4}$$

The SLBM system of Example 2.1 can now be used to illustrate the extended phased-mission model. The verbal description of that system's operation translates into the following mathematical structure:

$$\text{for phase } 0, \quad \phi_0(\underline{x}) = x_S x_C x_N$$

$$\text{for phase } 1, \quad \phi_1(\underline{x}) = x_S x_{FC} x_G$$

$$\text{for phase } 2, \quad \phi_2(\underline{x}) = x_S x_E$$

$$\text{for phase } 3, \quad \phi_3(\underline{x}) = [(x_{VP} x_{IP}) \vee (x_{VS} x_{IS})] x_G x_{RF}$$

$$\text{for phase } 4, \quad \phi_4(\underline{x}) = (x_{VP} \vee x_{VS}) x_G x_J x_{RS}$$

$$\text{for phase } 5, \quad \phi_5(\underline{x}) = x_W$$

The equivalent block-diagram representation of the system is shown in Figure 2.3.

A mathematical statement of mission reliability for this system results from substitution of the phase structure functions into Equation 2.2.4. Thus, mission reliability is

$$\begin{aligned} p = E\{ & [X_S(t_0)X_C(t_0)X_N(t_0)][X_S(t_1)X_{FC}(t_1)X_G(t_1)] \\ & [X_S(t_2)X_E(t_2)][X_{VP}(t_3)X_{IP}(t_3) \vee X_{VS}(t_3)X_{IS}(t_3)] \\ & [X_G(t_3)X_{RF}(t_3)][X_{VP}(t_4) \vee X_{VS}(t_4)] \\ & [X_G(t_4)X_J(t_4)X_{RS}(t_4)][X_W(t_5)]\} \end{aligned} \tag{2.2.5}$$

Evaluation of Equation 2.2.5 is not straightforward. Even though the components perform independently, the performance indicator variables for the same component at different times are obviously not independent. Hence it is not clear at this stage how to proceed. In the next section the transformation due to Esary and Ziehms [1975] will be used to convert expressions such as Equation 2.2.5 into the expectation of sums and

products of independent random variables. The cost associated with this procedure--that of a significant increase in the number of variables--will be made readily apparent when the system of Example 2.1 is transformed.

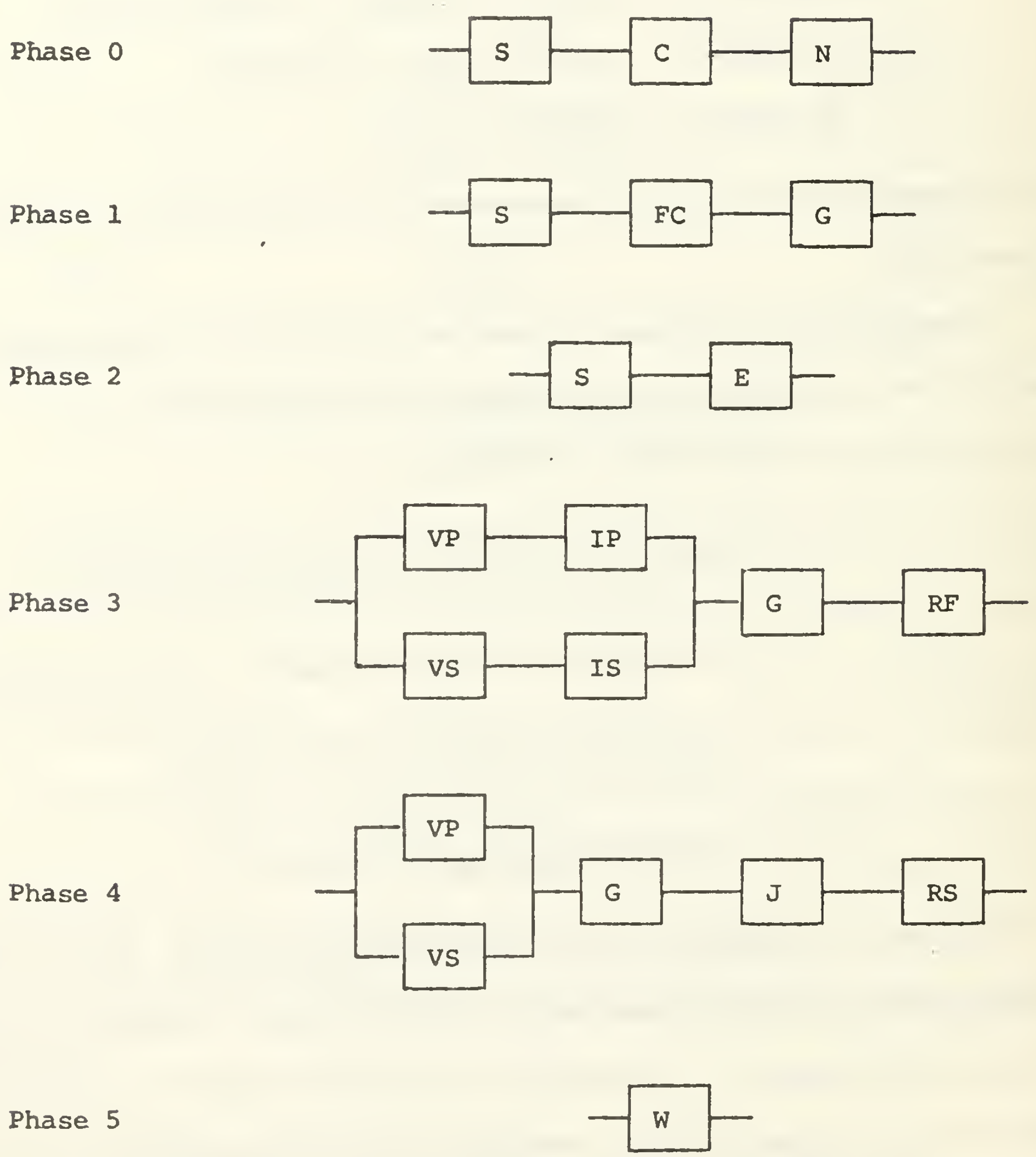


Figure 2.3. Phase configurations for the system of Example 2.1.

2.3 TRANSFORMATION OF THE MULTI-PHASE MISSION PROBLEM

In the context of the phased-mission problem under investigation, the transformation suggested by Esary and Ziehms [1975] consists of the following steps:

(a) Replace each component C_k in the configuration for phase j, $j=0,1,\ldots,m$, by a series arrangement of independent pseudo components $C_{k0},\ldots, C_{kj}$ with performance state indicator variables $U_{k0},\ldots, U_{kj}$, where

$$\alpha_k(t_0) = P[U_{k0} = 1] = P[X_k(t_0) = 1]$$

and, for $i=1,\ldots,j$,

$$\pi_{ki} = P[U_{ki} = 1] = P[X_k(t_i) = 1 | X_k(t_{i-1}) = 1]$$

(b) Connect the transformed phase configurations in series.

The result of this procedure is an *equivalent system* of at most $n(m+1)$ pseudo components which is coherent and performs a single-phase mission.

Since $P[X_k(t_j) = 1] = P[U_{k0}U_{k1}\cdots U_{kj} = 1]$, $X_k(t_j)$ has the same distribution as the product $U_{k0}U_{k1}\cdots U_{kj}$. ($X_k(t_j)$ is said to be stochastically equal to ($\overset{st}{=}$) $U_{k0}U_{k1}\cdots U_{kj}$.) It follows from Theorem 3.1 of Esary-Ziehms [1975] that

$$[X_k(t_0),X_k(t_1),\ldots,X_k(t_m)] \overset{st}{=} [U_{k0},U_{k0}U_{k1},\ldots,U_{k0}U_{k1}\cdots U_{km}]$$

and by independence of the components of the original system that

$$[\underline{X}(t_0),\underline{X}(t_1),\ldots, \underline{X}(t_m)] \overset{st}{=} [\underline{U}_0,\underline{U}_0\underline{U}_1,\ldots, \underline{U}_0\underline{U}_1\cdots \underline{U}_m]$$

where

$$\underline{U}_j = [U_{1j}, U_{2j},\ldots, U_{nj}]$$

and

$$\underline{U}_j\underline{U}_k = [U_{1j}U_{1k}, U_{2j}U_{2k},\ldots, U_{nj}U_{nk}]$$

Thus the reliability of the original system as given by Equation 2.2.4 is the same as the reliability of the equivalent system given by

(2.3.1) $$p = P\left\{\prod_{j=0}^{m} \phi_j[\underline{U}_0 \cdots \underline{U}_j] = 1\right\}$$

or more compactly as

(2.3.2) $$p = E\left\{\prod_{j=0}^{m} \phi_j[\underline{U}_0 \cdots \underline{U}_j]\right\}$$

The random variables in Equations 2.3.1 and 2.3.2 are mutually independent by construction, and hence computing the expected value is theoretically routine. Application of this transformation is illustrated in Figure 2.4,which shows the transformed phase configurations for the system of Example 2.1.

Figure 2.4 provides a graphic demonstration of the practical difficulty to be encountered when applying the transformation. The number of components in the equivalent system will likely be large for any moderately complex system. Even though computer algorithms for evaluation of block diagrams and fault trees are available, the computation time and memory requirements associated with such a large number of components would be excessive.

Rubin [1964] and Weisburg and Schmidt [1966] pointed out a technique which results in simplified configurations for the early phases of a phased mission. Esary and Ziehms [1975] provide justification for this procedure called *cut cancellation*. It is well known that every coherent system can be represented as a series structure of subsystems, each of which consists of the components belonging to a *minimal cut set* connected in parallel. (See, for example, Barlow and Proschan [1975a].) A cut set is a set of components which by all failing causes the system

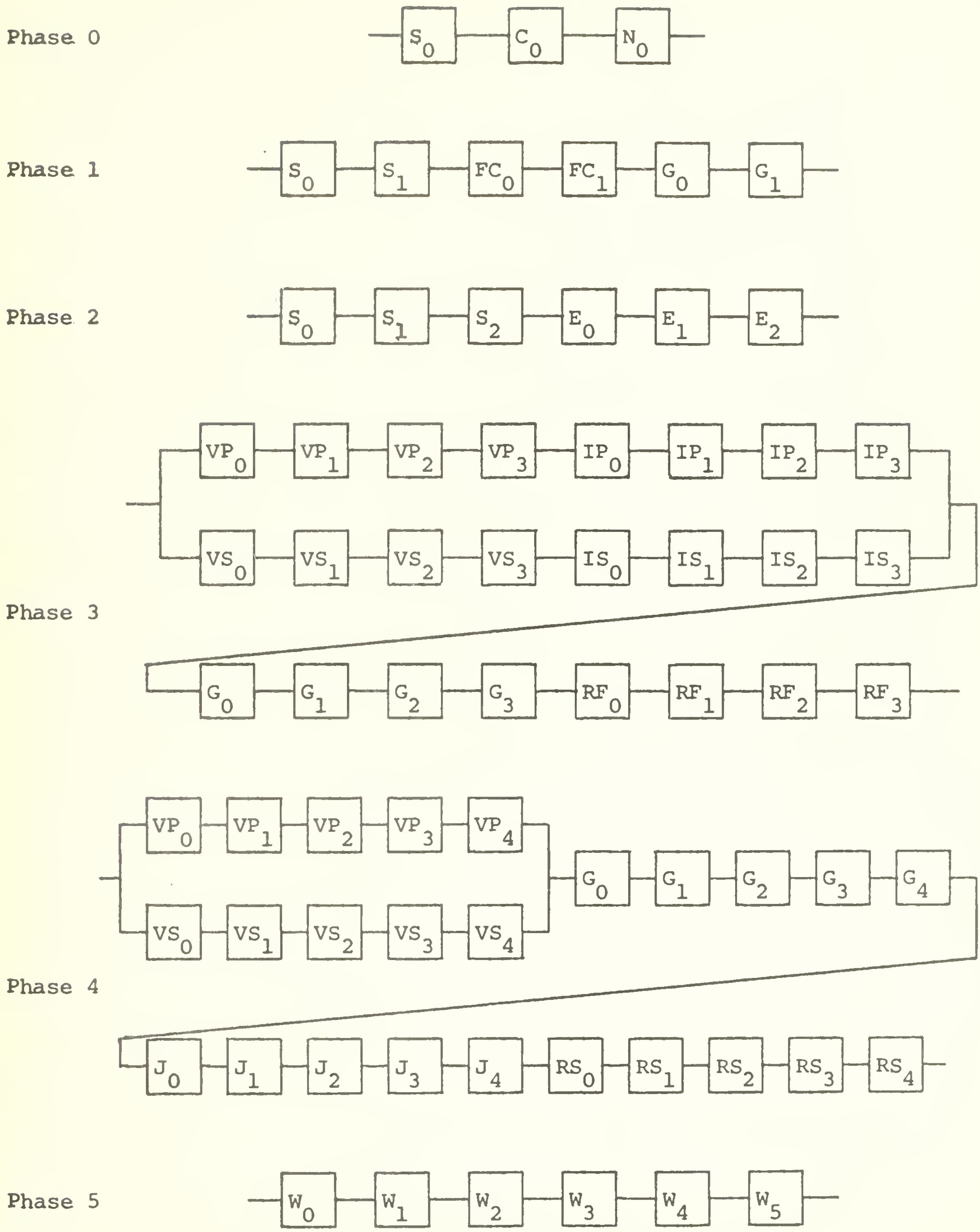


Figure 2.4. Transformed phase configurations for system of Example 2.1.

to fail. A minimal cut set is one which is no longer a cut set if any of the components are removed. The cut-cancellation procedure involves the following steps:

(a) Find the minimal cut sets for each phase.

(b) Remove from the list of minimal cut sets for any phase each minimal cut set which contains a minimal cut set for a later phase.

(c) Reconstitute the system from the remaining minimal cut sets. The reliability of the resulting configuration is unchanged from that of the original system. Performing cut cancellation does not, however, reduce the number of components in the equivalent system.

<u>Example</u> <u>2.3</u>. The minimal cut sets and indicated cancellations for the system of Example 2.1 are:

for phase 0, ~~{S}~~, {C}, {N}

for phase 1, ~~{S}~~, {FC}, ~~{G}~~

for phase 2, {S}, {E}

for phase 3, ~~{VP,VS}~~, {VP,IS}, {VS,IP}, {IP,IS}, ~~{G}~~, {RF}

for phase 4, {VP,VS}, {G}, {J}, {RS}

for phase 5, {W} □

Examination of Figure 2.4 suggests a procedure which does result in a reduction in the number of components in the equivalent system. Look, for example, at phase 5. The pseudo components $W_0, \ldots, W_5$ appear nowhere else in the system configuration. Thus it is not necessary to transform component W into $W_0, \ldots, W_5$ in order to gain independence among components of the equivalent system. Leaving W untransformed, however, would nullify one of the other appealing features of the transformation technique. It is completely natural to use conditional phase reliabilities to describe component performance in each active phase

and to keep separate the question of initial availability. A compromise which retains this aspect and yet reduces the number of components results from lumping pseudo components $W_0, \ldots, W_4$ together into a new pseudo component $W_{\vec{4}}$ where

$$P[U_{W\vec{4}} = 1] = P[U_{W4} = 1]P[U_{W3} = 1]P[U_{W2} = 1]P[U_{W1} = 1]P[U_{W0} = 1]$$

$$= P[X_W(t_4) = 1]$$

In general, a component is said to be *relevant* to system operation during any phase only if it appears in the configuration for that phase. If component C_k becomes relevant for the first time in phase j of a mission, a general rule for *component reduction* is to replace pseudo components $C_{k0}, \ldots, C_{k,j-1}$ with a new pseudo component $C_{k,\vec{j}-1}$ wherever they appear in the equivalent system. The new pseudo component has reliability given by

$$P[U_{k,\vec{j}-1} = 1] = \alpha_k(t_{j-1}) = \alpha_k(t_0)\pi_{k1} \cdots \pi_{k,j-1}$$

and the reliability of the reduced equivalent system is the same as that of the original system.

The equivalent system configuration for the SLBM system of Example 2.1 which results from the use of cut cancellation and component reduction is shown in Figure 2.5. In this case the number of pseudo components is reduced from 53 to 26.

In spite of efforts to reduce the number of components, many systems are so large and complex that direct calculation of mission reliability is infeasible. In these cases the upper and lower bounds on mission reliability contained in Ziehms [1975] can be applied directly to the model presented here. Ziehms provides a detailed development

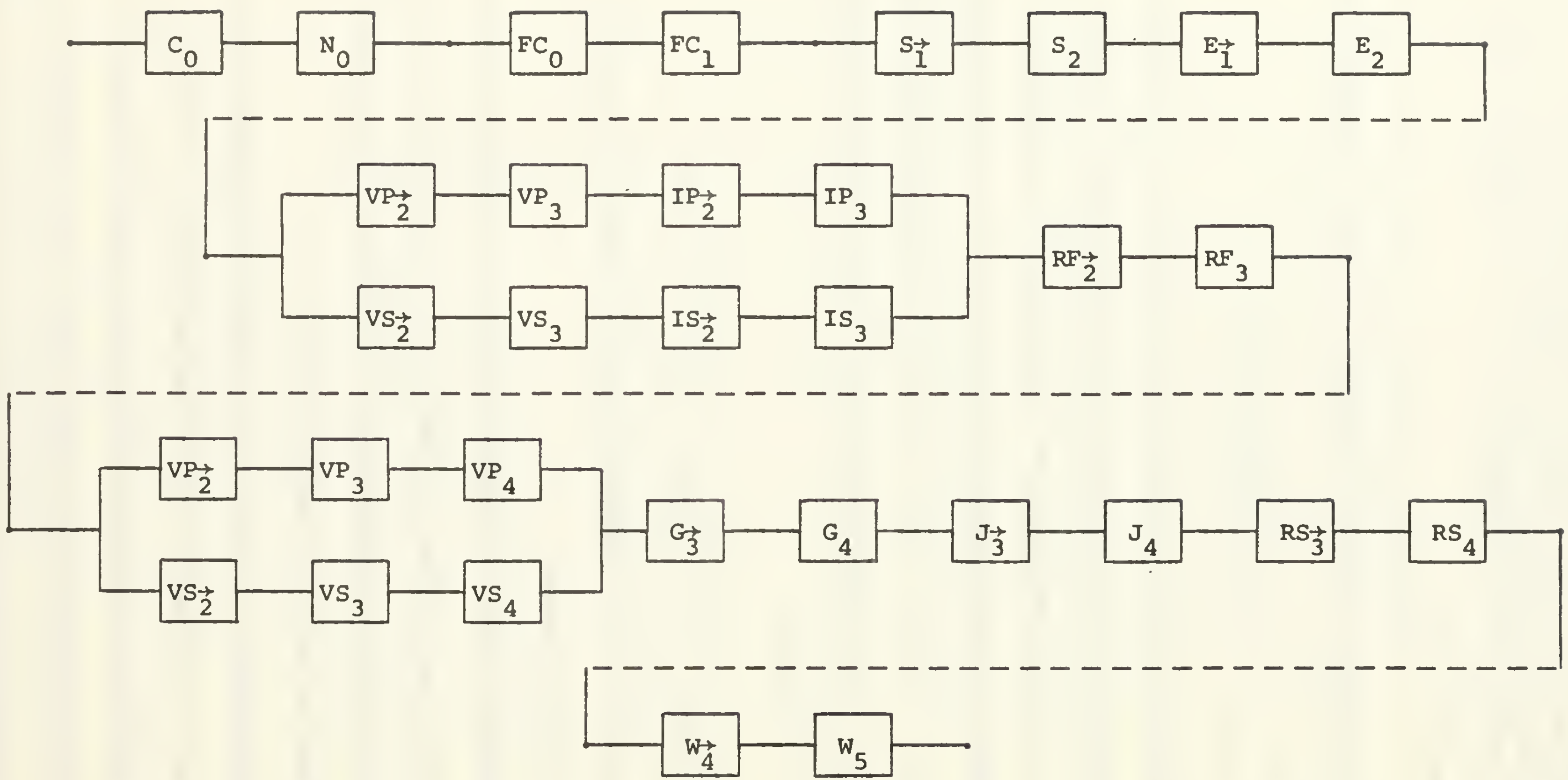


Figure 2.5. Equivalent system for Example 2.1 after component reduction and cut cancellation.

of these bounds and offers some criteria for choosing the best among them.

Before mission reliability or bounds thereon can be computed, the component availabilities and phase reliabilities must be calculated. This is the subject of the next section.

2.4 AVAILABILITIES AND PHASE RELIABILITIES

No attempt is made in this section to catalog existing models for component availabilities, nor are any new models developed. Standard models are used to illustrate a typical approach to the development of component availabilities. Barlow and Proschan [1975a] and Cox [1962] are good sources for additional details on this subject.

There are two different situations to be explored in connection with component availabilities. First is the issue of the availability of components which remain dormant during the O R phase (and thus are not maintained). Let the random variable T_k be the *active* lifelength of component C_k, that is, the time from t_0 until the component fails. Then, for all $t > 0$,

$$P[T_k > t] = P[T_k > t | T_k > 0] P[T_k > 0]$$

$$= \alpha_k(t_0) P[T_k > t | T_k > 0]$$

Thus $\alpha_k(t_0)$ is the probability that component C_k is available when first activated, and $[1 - \alpha_k(t_0)]$ is the probability that it will fail to operate because of a manufacturing defect, mishandling, or some other cause unrelated to service failure. Since failures of this type will generally be independent of the length of the O R phase, the argument t_0 can usually be dropped to yield the constant availability α_k.

The second case is that of repairable components. There are as many models for the performance processes of repairable components as there are different maintenance schemes; however unless it is assumed that components have exponentially-distributed times to failure, there are serious difficulties in accounting for the residual time to failure for those components functioning at time t_0. Thus here, as in most applications of reliability theory, constant component failure rates are assumed. It is convenient (but not necessary) to assume that component repair times are exponentially distributed as well. The resulting performance process for each component is an alternating renewal process, hereafter called the *exponential-exponential* performance process. Specifically, it is assumed that during the O R phase component C_k has a constant failure rate λ_{k0} and a constant repair rate μ_{k0}. A standard renewal theory argument (see Cox [1962] or Barlow and Proschan [1975a]) shows that if component C_k is functioning at time t=0, then its availability at a later time t is given by

(2.4.1) $$\alpha_k(t) = [\lambda_{k0} + \mu_{k0}]^{-1}\left(\mu_{k0} + \lambda_{k0}e^{-(\lambda_{k0}+\mu_{k0})t}\right)$$

and if it is down at time t=0 then

(2.4.2) $$\alpha_k(t) = \mu_{k0}[\lambda_{k0} + \mu_{k0}]^{-1}\left(1 - e^{-(\lambda_{k0}+\mu_{k0})t}\right)$$

If, as generally assumed, time t_0 is unknown, then Equations 2.4.1 and 2.4.2 are of little use in providing the required numerical value for $\alpha_k(t_0)$. Unless component availabilities are known from some other source (similar systems or testing programs) there is no analytic alternative to the use of bounds. The most common approach is to approximate $\alpha_k(t_0)$ by the (long-run) *availability* given by

$$\alpha_k = \lim_{t\to\infty} \alpha_k(t) = \mu_{k0}[\lambda_{k0} + \mu_{k0}]^{-1} \tag{2.4.3}$$

This is equivalent to assuming that the performance process is in equilibrium or steady state at time t=0, i.e. that $P[X_k(0) = 1] = \alpha_k$. It is easy to see that the availability α_k as given by Equation 2.4.3 is a lower bound on $\alpha_k(t)$ as long as $P[X_k(0) = 1] \geq \alpha_k$.

A better lower bound on $\alpha_k(t_0)$ is available if there is an upper limit τ on the duration of the O R phase. Since $\alpha_k(t)$ as given by Equation 2.4.1 is a decreasing function of time, $\alpha_k(t_0)$ is bounded below by

$$\underline{\alpha}_k = [\lambda_{k0} + \mu_{k0}]^{-1}\left(\mu_{k0} + \lambda_{k0} e^{-(\lambda_{k0}+\mu_{k0})\tau}\right) \tag{2.4.4}$$

provided component C_k is functioning at time t=0.

Example 2.4. Consider the navigation component of the SLBM system of Example 2.1, and suppose it is subject to failure at rate λ_{N0} and repair at rate μ_{N0} during the O R phase. Assume that the submarine patrols for a maximum of τ days. Upon completion of its patrol, the submarine is relieved on station by another of the same type and returns to its home base for an upkeep period, during which all repairable components are restored to working condition. Then a conservative estimate of the availability of the navigation component at time t_0 is given by Equation 2.4.4. □

Other bounds of this type can be found when there is random selection of the component initial state. As long as the availability of component C_k at time t=0 is at least α_k, $\alpha_k(t)$ is decreasing with time, and $\alpha_k(\tau)$ provides a lower bound. Otherwise, the initial availability is a lower bound on $\alpha_k(t_0)$.

It is not unrealistic to assume that every component has a constant failure rate within each phase. In this case the phase reliabilities, which are the final ingredients needed to perform calculations, take on a particularly simple form. Let the failure rate of component C_k in active phase j be λ_{kj}, k=1,...,n; j=1,...,m. Then the conditional phase reliability is given by

$$\pi_{kj} = P[X_k(t_j) = 1 | X_k(t_{j-1}) = 1] = e^{-\lambda_{kj} d_j}$$

$$k=1,\ldots,n;\ j=1,\ldots,m.$$

After the component availabilities and conditional phase reliabilities have been determined, the final step is calculation of mission reliability. This chapter is concluded with a brief sketch of this step for the SLBM system of Example 2.1. Similar calculations are shown in much greater detail for a less complex system in Example 3.2.

Assume that the initial availability and conditional phase reliabilities have been determined for each component of the SLBM system. The statement of mission reliability can be written down in reduced form (after cut cancellation and component reduction) directly from Figure 2.5. Thus mission reliability is given by

$$\begin{aligned} p \;=\; & E\{U_{C0}U_{N0}U_{FC0}U_{FC1}U_{\vec{S1}}\, U_{S2}U_{\vec{E1}}\, U_{E2} \\ (2.4.5) \qquad & \times\ [(U_{\vec{VP2}}\, U_{VP3}U_{\vec{IP2}}\, U_{IP3}) \vee (U_{\vec{VS2}}\, U_{VS3}U_{\vec{IS2}}\, U_{IS3})] \\ & \times\ U_{\vec{RF2}}\, U_{RF3}[(U_{\vec{VP2}}\, U_{VP3}U_{VP4}) \vee (U_{\vec{VS2}}\, U_{VS3}U_{VS4})] \\ & \times\ U_{\vec{G3}}\, U_{G4}U_{\vec{J3}}\, U_{J4}U_{\vec{RS3}}\, U_{RS4}U_{\vec{W4}}\, U_{W5}\} \end{aligned}$$

Let Π_{kj} be the *unconditional* reliability of component C_k at the end of

phase j where

$$\Pi_{kj} = \alpha_k(t_0)\pi_{k1}\cdots\pi_{kj}$$

Then, after expanding Equation 2.4.5 and performing idempotent cancellations ($U_{kj}U_{kj} = U_{kj}$), the mission reliability as a function of component availabilities and reliabilities is

$$\begin{aligned} p = {} & \alpha_C(t_0)\alpha_N(t_0)\Pi_{FC1}\Pi_{S2}\Pi_{E2}\Pi_{RF3}\Pi_{G4}\Pi_{J4}\Pi_{RS4}\Pi_{W5} \times \\ & \{\Pi_{VP3}\Pi_{VS3}(\Pi_{IP3}\pi_{VS4}(1-\pi_{VP4}) + \Pi_{IS3}\pi_{VP4}(1-\pi_{VS4}) \\ & - \Pi_{IP3}\Pi_{IS3}(\pi_{VP4} + \pi_{VS4} - \pi_{VP4}\pi_{VS4}) \\ & + \Pi_{VP4}\Pi_{IP3} + \Pi_{VS4}\Pi_{IS3}\} \end{aligned}$$

Extension of the model presented in this chapter to the case of a multiple-objective mission is discussed in Chapter 3. The methods for bounding component availabilities and calculating phase reliabilities discussed in this section are equally relevant there.

3. MULTIPLE OBJECTIVE MISSIONS

The performance of systems having more than one objective is considered in this chapter. There are obviously many ways in which a mission statement could be written to recognize multiple objectives. Here, the investigation is limited to those cases in which the objectives are all of approximately the same importance or rank. Further, it is assumed that all objectives are of the same type--that is, are in some sense repetitive in nature. Even with these restrictions, there remains too much latitude in system organization and performance characteristics to permit the development of a universally applicable model. The mathematical model developed in the following sections, which is motivated by the Fleet Ballistic Missile system, is one particularization. Nevertheless, the approach is sufficiently general to allow its adaptation to other situations.

3.1 TEMPORAL STRUCTURE OF THE MULTI-OBJECTIVE MISSION

The system to be considered is assumed to have a multi-phase mission as before. The mission consists of $r \geq 1$ objectives, each containing several tasks--one per phase. The performance of the tasks associated with any one of the objectives involves the use of some components which must also be used (either simultaneously or at another time) in the performance of tasks associated with other objectives. It is assumed that associated with each objective is a subset of components which are used only in the performance of tasks related to that objective. Components associated with more than one objective will be said to comprise the *master system* and components unique to objective i, $i=1,\ldots,r$, will

make up the i^{th} *subsystem*. All objectives are assumed to have the same structure. Thus the block diagram (or structure function) for the j^{th} phase of objective i is the same as that for the j^{th} phase of any other objective (but may involve physically different components.

The time sequence of phases is shown in Figure 3.1. Those phases whose configurations involve master system components are depicted on the horizontal time line and are called *trunk phases*. Those phases in which only subsystem i components are relevant are shown on the i^{th} vertical time line and are called *branch i phases*. The sequence of trunk phases consists of

--the O R phase, shared in common by all objectives and involving only master system components;

--a active phases shared in common by all objectives, involving only master system components;

--b phases associated with objective 1, in which master system and subsystem 1 components are relevant;

--b phases associated with objective 2, in which master system and subsystem 2 components are relevant;

⋮

--b phases associated with objective r, in which master system and subsystem r components are relevant.

Each branch consists of c-b phases so that for each objective there are a total of a+c+1 phases.

The trunk phases shared in common by all objectives are denoted F_{oj}, j=0,1,...,a, and those phases unique to objective i, i=1,...,r, are denoted successively by F_{ij}, j=1,...,c.

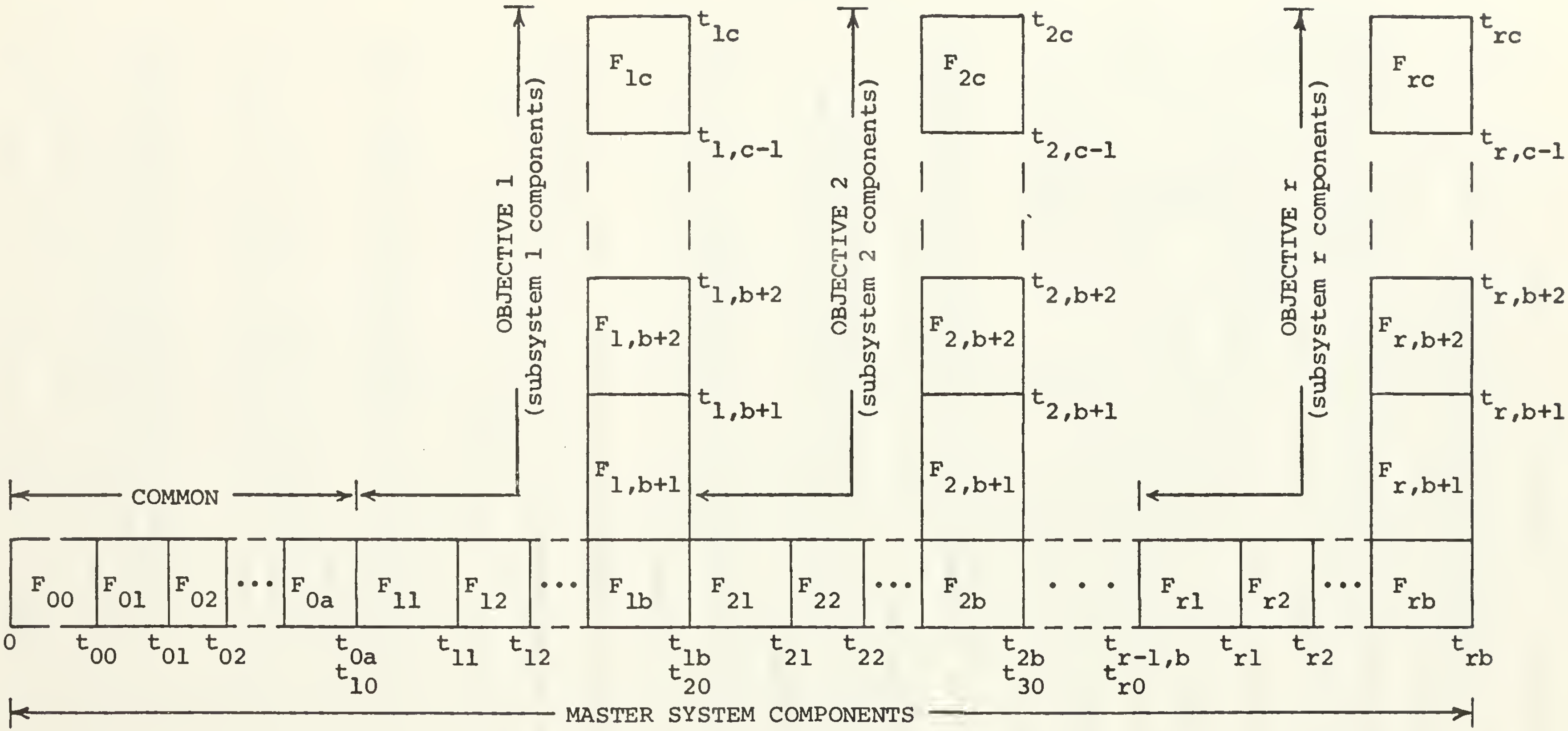


Figure 3.1. Multi-objective mission phase sequence

The time labels displayed in Figure 3.1 are based on the following conventions:

TIME	EVENT
t_{0j}, $j=0,\ldots,a-1$	Phase F_{0j} ends and phase $F_{0,j+1}$ begins.
$t_{0a} = t_{10}$	Phase F_{0a} ends and phase F_{11} begins.
t_{ij}, $i=1,\ldots,r;\ j=1,\ldots,c-1$	Phase F_{ij} ends and phase $F_{i,j+1}$ begins.
t_{ic}, $i=1,\ldots,r$	Phase F_{ic} ends.
$t_{ib} = t_{i+1,0}$, $i=1,\ldots,r$	Phase F_{ib} ends and phase $F_{i+1,1}$ begins.
t_{rb}	Phase F_{rb} ends.

For objective i to be successfully completed, the system must be available at the end of phase F_{00} (O R phase), and it must function satisfactorily throughout phases $F_{01}, \ldots, F_{0a}$, and phases $F_{i1}, \ldots, F_{ic}$. System performance in other phases is irrelevant to this objective. Aside from the changes in phase arrangement, this is precisely the problem considered in Chapter 2, and the methods presented there could be used to calculate the probability of successfully completing objective i, $i=1,\ldots,r$. Because of the obvious dependencies among the objectives, some additional mathematical structure is required to support joint probability statements about two or more of the objectives. This structure is developed in the following section.

3.2 MATHEMATICAL MODEL OF THE MULTI-OBJECTIVE MISSION

The master system is assumed to have n components, $C_1, \ldots, C_n$, and subsystem i to have m components, $D_1^{(i)}, \ldots, D_m^{(i)}$, $i=1,\ldots, r$. The master system performance state indicator vector at time t is

$$\underline{X}(t) = [X_1(t), \ldots, X_n(t)]$$

and, for $i=1,\ldots,r$, the performance state indicator vector for subsystem i at time t is

$$\underline{Y}^{(i)}(t) = [Y_1^{(i)}(t), \ldots, Y_m^{(i)}(t)].$$

A vector of objective indicator variables $\underline{J} = [J_1, \ldots, J_r]$ is defined by

$$J_i = \begin{cases} 1 \text{ if objective } i \text{ is successful,} \\ 0 \text{ otherwise.} \end{cases}$$

The O R phase is assumed to have a semi-coherent structure function ψ_0, phase F_{0j}, $j=1,\ldots,a$, to have a coherent structure function ψ_j, and phase F_{ij}, $i=1,\ldots,r$; $j=1,\ldots,c$, to have a coherent structure function ϕ_j. For notational convenience, let $X_k(t_{ij}) = X_{kij}$, $Y_k^{(i)}(t_{ij}) = Y_{kij}$, $\underline{X}(t_{ij}) = \underline{X}_{ij}$, and $\underline{Y}^{(i)}(t_{ij}) = \underline{Y}_{ij}$. Then the probability of successfully completing objective i, $i=1,\ldots,r$, is given by

$$(3.2.1) \qquad p_i = P\{\prod_{j=0}^{a}\psi_j(\underline{X}_{0j}) \prod_{j=1}^{b}\phi_j(\underline{X}_{ij}, \underline{Y}_{ij}) \prod_{j=b+1}^{c}\phi_j(\underline{Y}_{ij}) = 1\}.$$

Thus, for $i=1,\ldots, r$,

$$(3.2.2) \qquad J_i \overset{st}{=\!=} \prod_{j=0}^{a}\psi_j(\underline{X}_{0j}) \prod_{j=1}^{b}\phi_j(\underline{X}_{ij}, \underline{Y}_{ij}) \prod_{j=b+1}^{c}\phi_j(\underline{Y}_{ij}) .$$

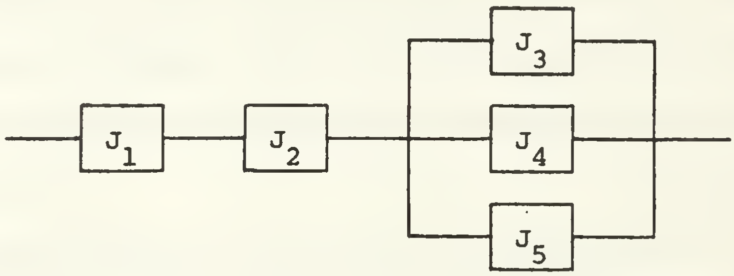
In Chapter 2, the system was required to function satisfactorily in all active phases in order for the mission to be successful. Thus, after each phase structure had been transformed, the resulting structures were connected in series to form the equivalent system. It seems natural, then, to consider a generalization of this procedure which permits the transformed phase structures to be connected in other configurations. Such an arrangement is appropriate in the case of the multi-objective mission.

Clearly, the transformed structures for the phases associated with any single objective should still be connected in series; however only when mission success is defined as successful completion of all objectives should the phase structures associated with one objective be connected in series with those for all other objectives. A link between mission success and success in each of the objectives is required so that the method of connecting transformed phase structures can be prescribed. Accordingly, it is assumed that mission success is completely determined by the outcomes on the r objectives and that there is a binary function η of the binary random variables $J_1, \ldots, J_r$ defined by

$$\eta[J_1, \ldots, J_r] = \begin{cases} 1 \text{ if the mission is successful,} \\ 0 \text{ otherwise.} \end{cases}$$

It is further assumed that this *mission success structure function* is coherent. Then mission success can be represented pictorially as a block diagram in which the "components" are objectives. This is a mild assumption since most measures of mission success will be increasing functions whose ranges can be partitioned into regions defining success and failure which can then be re-scaled to satisfy the requirements for coherence. The following example illustrates this concept.

Example 3.1. A multi-component system performs a mission which has five objectives. The mission is considered a success if objectives 1 and 2 and at least one of the remaining objectives are accomplished. Then the mission success block diagram is



□

In those instances when a mission success structure is not or can not be specified, artificial mission success structures can be used to obtain other quantities of interest. In such cases it may be particularly useful to obtain the probability distribution which governs objective accomplishment. When, as assumed in this chapter, objectives are of the same type and carry the same rank, it is sufficient to determine the distribution of the number of objectives accomplished during the mission. If N is a random variable representing the number of objectives accomplished and η_k is the k out of r structure function, k=1,...,r, then

$$P[N \geq k] = P[\eta_k(\underline{J}) = 1] = E[\eta_k(\underline{J})], \; k=1,\ldots,r.$$

Thus the probability distribution of N is given by

$$(3.2.3) \qquad \begin{aligned} P[N = 0] &= 1 - E[\eta_1(\underline{J})], \\ P[N = k] &= E[\eta_k(\underline{J})] - E[\eta_{k+1}(\underline{J})], \; k=1,\ldots,r-1, \\ P[N = r] &= E[\eta_r(\underline{J})]. \end{aligned}$$

It may frequently be useful to summarize system performance by specifying the expected number of successes among the r objectives.

This expected value can be found using the k out of r structure functions or directly from the objective success probabilities, since

$$E[N] = \sum_1^r P[N \geq k] = \sum_1^r E[\eta_k(\underline{J})] = \sum_1^r E[J_k] \; . \tag{3.2.4}$$

Once the mission success structure function to be used is specified, the mathematical description of the mission is complete. If the structure function is based on actual mission requirements (rather than being a device to obtain other quantities) then the probability of mission success is given by

$$p = P[\eta(J_1,\ldots, J_r) = 1]. \tag{3.2.5}$$

Substituting for $J_1,\ldots, J_r$ from Expression 3.2.2 yields the probability of mission success in terms of the component performance indicator variables. Of course the same procedure is appropriate even when the structure function η is just an intermediate device, but the resulting expression is not the probability of mission success.

As was the case in Chapter 2, the expression for the mission success probability is the expected value of a combination of component performance indicator variables which are not mutually independent, since a component's state at one time point is correlated with its state at another. The next section provides the details of the transformation which yields the probability of mission success as the expected value of a function of mutually independent random variables.

3.3 TRANSFORMATION OF THE MULTI-OBJECTIVE MISSION

The procedure for transforming the system with a multi-objective mission into one with an equivalent single-objective, single-phase mission is essentially the same as that presented in Chapter 2. Some of the details are changed because of the modified layout of the mission; however the basic concept remains that of breaking each component of the system into a series of pseudo components. The transformation procedure consists of the following steps:

(a) Replace master system component C_k, $k=1,\ldots,n$, in phase F_{ij} by a series system of pseudo components C_{k00}, ..., C_{kij}, $i=0$, $j=0,\ldots,a$; $i=1,\ldots,r$, $j=1,\ldots,c$, which perform independently.

(b) Replace component $D_k^{(i)}$ of the $i^{\underline{th}}$ subsystem in every phase F_{ij} in which it appears by a series configuration of independent pseudo components D_{ki0}, ..., D_{kij}, $i=1,\ldots,r$, $j=1,\ldots,c$, $k=1,\ldots,m$.

(c) Connect the transformed phase structures for all phases associated with objective i in series to form the equivalent system for objective i.

(d) Connect the equivalent systems for all objectives in the manner prescribed by the mission success structure function.

To see that this transformation procedure yields an equivalent system having a single-objective, single-phase mission and the same mission success probability as the original system, performance state indicator variables for the pseudo components must be introduced. For $k=1,\ldots,n$, let $U_{k00},\ldots, U_{kij}$ be independent performance state indicator variables for pseudo components $C_{k00},\ldots, C_{kij}$ with

$$P[U_{k00} = 1] = P[X_{k00} = 1]$$

and, for $i=0$, $j=1,\ldots,a$ and $i=1,\ldots,r$, $j=1,\ldots,b$,

$$P[U_{kij} = 1] = P[X_{kij} = 1 | X_{ki,j-1} = 1].$$

For $k=1,\ldots,m$ and $i=1,\ldots,r$, let $V_{ki0}, \ldots, V_{kij}$ be independent performance state indicator variables for pseudo components $D_{ki0}, \ldots, D_{kij}$ with

$$P[V_{ki0} = 1] = P[Y_{ki0} = 1] \text{ and}$$

$$P[V_{kij} = 1] = P[Y_{kij} = 1 | Y_{ki,j-1} = 1],\ j=1,\ldots,c.$$

It is immediate that $X_{kij} \overset{st}{=} U_{k00} \cdots U_{kij}$ and that $Y_{kij} \overset{st}{=} V_{ki0} \cdots V_{kij}$. Further, from Theorem 3.1 of Esary-Ziehms [1975], it follows that for $k=1,\ldots,n$,

$$[X_{k00}, X_{k01}, \ldots, X_{krb}] \overset{st}{=} [U_{k00}, U_{k00}U_{k01}, \ldots, U_{k00}U_{k01} \cdots U_{krb}]$$

and, for $i=1,\ldots,r$ and $k=1,\ldots,m$, that

$$[Y_{ki0}, Y_{ki1}, \ldots, Y_{kic}] \overset{st}{=} [V_{ki0}, V_{ki0}V_{ki1}, \ldots, V_{ki0}V_{ki1} \cdots V_{kic}].$$

Then, since the original components perform independently,

$$(3.3.1)\qquad \begin{aligned} [\underline{X}_{00}, \underline{X}_{01}, \ldots, \underline{X}_{rb}] &\overset{st}{=} [\underline{U}_{00}, \underline{U}_{00}\underline{U}_{01}, \ldots, \underline{U}_{00}\underline{U}_{01} \cdots \underline{U}_{rb}] \\ [\underline{Y}_{i0}, \underline{Y}_{i1}, \ldots, \underline{Y}_{ic}] &\overset{st}{=} [\underline{V}_{i0}, \underline{V}_{i0}\underline{V}_{i1}, \ldots, \underline{V}_{i0}\underline{V}_{i1} \cdots \underline{V}_{ic}], \end{aligned}$$

$i=1,\ldots,r$, where

$$\underline{U}_{ij} = [U_{1ij}, U_{2ij}, \ldots, U_{nij}]$$

$$\underline{V}_{ij} = [V_{1ij}, V_{2ij}, \ldots, V_{mij}]$$

$$\underline{U}_{ij}\underline{U}_{k\ell} = [U_{1ij}U_{1k\ell}, U_{2ij}U_{2k\ell}, \ldots, U_{nij}U_{nk\ell}]$$

$$\underline{V}_{ij}\underline{V}_{i\ell} = [V_{1ij}V_{1i\ell}, V_{2ij}V_{2i\ell}, \ldots, V_{mij}V_{mi\ell}] .$$

Further, any vector created by combining the left-hand sides of Equations 3.3.1 is stochastically equal to the same arrangement of the right-hand sides. Substituting into Equation 3.2.2 yields the result:

$$J_i \overset{st}{=} \prod_{j=0}^{a} \psi_j(\underline{U}_{00}\cdots\underline{U}_{0j}) \prod_{j=1}^{b} \phi_j(\underline{U}_{00}\cdots\underline{U}_{ij}, \underline{V}_{i0}\cdots\underline{V}_{ij}) \times \prod_{j=b+1}^{c} \phi_j(\underline{V}_{i0}\cdots\underline{V}_{ij}), \ i=1,\ldots,r. \tag{3.3.2}$$

For $i=1,\ldots,r$, let $\Phi_{ij} = \phi_j(\underline{U}_{00}\cdots \underline{U}_{ij}, \underline{V}_{i0}\cdots \underline{V}_{ij})$, $j=1,\ldots,b$, and $\Phi_{ij} = \phi_j(\underline{V}_{i0}\cdots \underline{V}_{ij})$, $j=b+1,\ldots,c$, and let $\Psi_{0j} = \psi_j(\underline{U}_{00}\cdots \underline{U}_{0j})$, $j=0,\ldots,a$. Then Equation 3.3.2 can be written more compactly as

$$J_i \overset{st}{=} \prod_{j=0}^{a} \Psi_{0j} \prod_{j=1}^{c} \Phi_{ij}. \tag{3.3.3}$$

Finally, the reliability of the equivalent system for its single-objective, single-phase mission as given by

$$p = P\left\{\eta\left(\prod_{j=0}^{a}\Psi_{0j} \prod_{j=1}^{c}\Phi_{1j}, \ldots, \prod_{j=0}^{a}\Psi_{0j} \prod_{j=1}^{c}\Phi_{rj}\right) = 1\right\} \tag{3.3.4}$$

is equal to the probability of success in the original mission given by Equation 3.2.5.

The major benefit of the transformation which results in the equivalent system whose reliability is given by Equation 3.3.4 is the elimination of dependencies among the performance state indicator variables. Thus the probability of mission success can be obtained as the expected value of sums, products, and differences of independent random variables--a task which is conceptually straightforward. The procedure suffers from the same drawback as the transformation of Chapter 2, however,

since the number of pseudo components in the equivalent system is likely to be quite large. Some techniques for reducing the complexity of the equivalent system are discussed along with approximation methods in the next section.

3.4 SIMPLIFICATIONS AND APPROXIMATIONS

The cut cancellation procedure (appropriately modified) and the component reduction technique, which were discussed in Chapter 2, can also be applied to the multi-objective mission problem. The change in the cut cancellation method amounts to limiting the cancellations to only those minimal cut sets which contain a minimal cut set for a later phase *associated with the same objective*. Thus the step-by-step procedure for the multi-objective mission problem as formulated in this chapter is:

(a) Find the minimal cut sets for each phase associated with objective 1. (The minimal cut sets for the phases of any other objective i are the same with component $D_k^{(1)}$ replaced by $D_k^{(i)}$.)

(b) Remove from the list of minimal cut sets for phase F_{0j}, $j=0,\ldots,a$, each minimal cut set which contains a minimal cut set for phase $F_{0\ell}$, $\ell=j+1,\ldots,a$ or phase F_{1k}, $k=1,\ldots,c$.

(c) Remove from the list of minimal cut sets for phase F_{1j}, $j=1,\ldots,c-1$, each minimal cut set which contains a minimal cut set for phase F_{1k}, $k=j+1,\ldots,c$, and remove the corresponding minimal cut sets from the list for phase $F_{\ell j}$, $\ell=2,\ldots,r$.

(d) Reconstitute the system from the remaining minimal cut sets.

It follows from the proof of Remark 4.2 of Esary-Ziehms [1975] that this cut cancellation procedure does not affect the probability of

mission success.

Some component reductions have been incorporated into the transformation of Section 3.3. A potentially large number of pseudo components has been eliminated by automatically leaving component $C_k^{(i)}$, $k=1,\ldots,m$; $i=1,\ldots,r$, untransformed over the period from time $t=0$ until time t_{i0}, during which the component is irrelevant to system operation. Additional reductions will be possible in most applications. Thus if master system component C_k first becomes relevant to system operation in phase F_{ij} then the series arrangement of pseudo components $C_{k00},\ldots,C_{ki,j-1}$ can be replaced wherever it appears in the equivalent system by a single pseudo component $C_{ki,\vec{j}-1}$ with performance state indicator variable $U_{ki,\vec{j}-1}$, where

$$P[U_{ki,\vec{j}-1} = 1] = P[X_{ki,j-1} = 1] = P[U_{k00} \cdots U_{ki,j-1} = 1].$$

If subsystem i component $D_k^{(i)}$ first becomes relevant in phase F_{ij} then the series configuration of pseudo components $D_{ki0},\ldots,D_{ki,j-1}$ can be replaced wherever it appears in the equivalent system by the pseudo component $D_{ki,\vec{j}-1}$ with performance state indicator variable $V_{ki,\vec{j}-1}$ where

$$P[V_{ki,\vec{j}-1} = 1] = P[Y_{ki,j-1} = 1] = P[V_{ki0} \cdots V_{ki,j-1} = 1].$$

Although component reduction is a worthwhile technique, it cannot always be expected to reduce the number of pseudo components in the equivalent system to a manageable level. The last analytical resort when the number of pseudo components is too large to permit direct calculations is to approximate or bound the mission success probability.

The series organization of transformed phase structures, which was appropriate in Chapter 2 and in the work of Ziehms [1975], led to convenient upper and lower bounds on mission reliability. Unfortunately the general nature of the mission success structure η rules out bounds based on either phase reliabilities or objective success probabilities. It is tempting to try to bound the mission success probability from below by

$$p = E[\eta(J_1,\ldots,J_r)] \geq \eta(EJ_1,\ldots,EJ_r)$$

because this inequality holds when η is a series structure function. This is *not true*, however, since it is well known that the direction of the inequality is reversed when η is a parallel structure function. When η is other than a series or parallel structure function, such an inequality does not usually exist.

Although less convenient, it is still possible to place upper and lower bounds on the mission success probability by finding the minimal cut sets and minimal path sets of the equivalent system as a whole. (A minimal path set is a minimal set of components which by all functioning cause the system to function.) Then the minimal cut lower and minimal path upper bounds due to Esary and Proschan [1963] can be used to bound the reliability of the equivalent system. (See Barlow and Proschan [1975a] for a development of these bounds.)

It should be noted that in order to use Equations 3.2.3 to obtain the distribution of the number of objectives accomplished, Equation 3.3.4 must be computed exactly for each structure function $\eta_1,\ldots,\eta_r$. Any ordering established by the bounds discussed above would be destroyed by the subtraction required in Equations 3.2.3. Such is not the case,

however, when using Equation 3.2.4 to find the expected number of objectives accomplished. In this case the orderings are maintained so that lower bounds on $E[\eta_k(\underline{J})]$ or $E[J_k]$, $k=1,\ldots,r$, yield a lower bound on $E[N]$ and upper bounds on $E[\eta_k(\underline{J})]$ or $E[J_k]$, $k=1,\ldots,r$, yield an upper bound on $E[N]$.

3.5 EXAMPLE

It was stated at the beginning of this chapter that the multi-objective mission problem formulation was motivated by the Navy's Fleet Ballistic Missile system. The SLBM system of Example 2.1, which can be viewed as a hypothetical version of the FBM system, can be extended to provide an illustration of the basic notions of this chapter. Let the submarine of the SLBM system now carry r missiles. Then the system can be viewed as having r objectives--each of which is the destruction of a designated target. The master system would consist of those components which remain aboard the submarine, and the components of each missile would make up one subsystem. Those phases of the mission which take place aboard the submarine would be trunk phases, and branch i phases would commence when the $i^{\underline{th}}$ missile is launched.

This extended version of the SLBM system, while ideally suited for illustrating the basic elements of the multi-objective mission problem, is somewhat more complex than necessary for the purpose of demonstrating the mechanics of the transformation, cut-cancellation, and component-reduction procedures. The following example introduces a very simple system and then tracks it through the formulation and computation steps discussed in the earlier sections of this chapter.

After cut cancellation, the phases of objective i become

$$
\begin{array}{ccccccccccc}
-\boxed{C_1}- & \vdots & — & \vdots & — & \vdots & -\boxed{C_2}- & \vdots & — & \vdots & -\boxed{D_1^{(i)}}-\boxed{D_2^{(i)}}- \\
F_{00} & & F_{01} & & F_{i1} & & F_{i2} & & F_{i3} & & F_{i4}
\end{array}
$$

After transformation and incorporation of component reduction, the equivalent structures for the objectives are:

for objective 1,

$$- C_{100} - C_{21\vec{1}} - C_{212} - D_{11\vec{3}} - D_{114} - D_{21\vec{3}} - D_{214} -$$

for objective 2,

$$- C_{100} - C_{21\vec{1}} - C_{212} - C_{221} - C_{222} - D_{12\vec{3}} - D_{124} - D_{22\vec{3}} - D_{224} -$$

for objective 3,

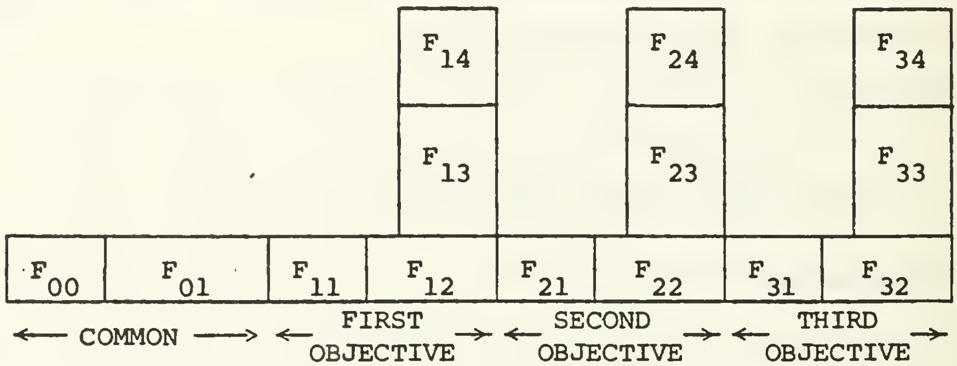
$$- C_{100} - C_{21\vec{1}} - C_{212} - C_{221} - C_{222} - C_{231} - C_{232} - D_{13\vec{3}} - D_{134} - D_{23\vec{3}} - D_{234} -$$

Thus,

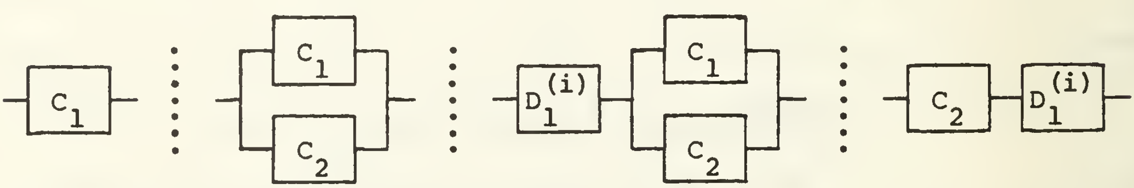
$$
\begin{aligned}
J_1 &\overset{st}{\equiv} U_{100}U_{21\vec{1}}\, U_{212}V_{11\vec{3}}\, V_{114}V_{21\vec{3}}\, V_{214} \\
J_2 &\overset{st}{\equiv} U_{100}U_{21\vec{1}}\, U_{212}U_{221}U_{222}V_{12\vec{3}}\, V_{124}V_{22\vec{3}}\, V_{224} \qquad (3.5.1) \\
J_3 &\overset{st}{\equiv} U_{100}U_{21\vec{1}}\, U_{212}U_{221}U_{222}U_{231}U_{232}V_{13\vec{3}}V_{134}V_{23\vec{3}}V_{234}
\end{aligned}
$$

Instead of specifying a particular mission success structure for this example, the distribution and expected value of the number of objectives accomplished, N, will be obtained. The structures needed to obtain the distribution are η_1, the 1 out of 3 structure; η_2, the 2 out of 3 structure; and η_3, the 3 out of 3 structure, whose block diagrams are

Example 3.2. A system with a three-objective mission has eight components. Components C_1 and C_2 make up the master system, and components $D_1^{(i)}$ and $D_2^{(i)}$ make up subsystem i, i=1,2,3. Each objective entails the successful completion of an O R phase and five active phases as shown in the following diagram.

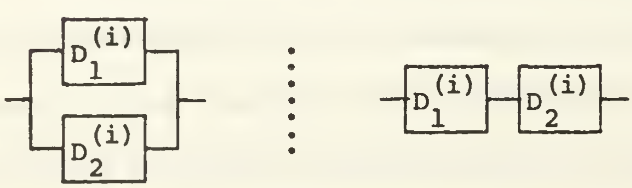


Thus, for this example, n=2, m=2, a=1, b=2, and c=4. The block diagrams for the trunk phases are

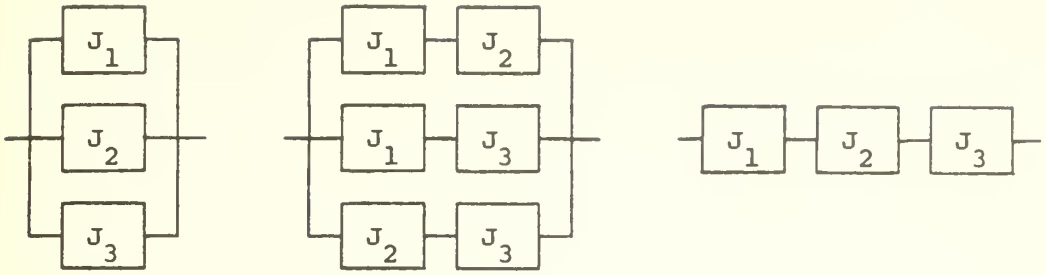


F_{00} F_{01} F_{i1}, i=1,2,3 F_{i2}, i=1,2,3

and those for the branch phases are



F_{i3}, i=1,2,3 F_{i4}, i=1,2,3



η_1 η_2 η_3

Computationally,

$$\eta_1(\underline{J}) = J_1 \vee J_2 \vee J_3$$

$$\doteq J_1 + J_2 + J_3 - J_1J_2 - J_1J_3 - J_2J_3 + J_1J_2J_3$$

(3.5.2) $$\eta_2(\underline{J}) = (J_1J_2) \vee (J_1J_3) \vee (J_2J_3)$$

$$= J_1J_2 + J_1J_3 + J_2J_3 - 2J_1J_2J_3$$

$$\eta_3(\underline{J}) = J_1J_2J_3$$

Then, from Equations 3.2.3, the distribution of N is given by

$$P[N=0] = 1 - E[J_1 + J_2 + J_3 - J_1J_2 - J_1J_3 - J_2J_3 + J_1J_2J_3]$$

$$P[N=1] = E[J_1 + J_2 + J_3 - 2J_1J_2 - 2J_1J_3 - 2J_2J_3 + 3J_1J_2J_3]$$

(3.5.3)

$$P[N=2] = E[J_1J_2 + J_1J_3 + J_2J_3 - 3J_1J_2J_3]$$

$$P[N=3] = E[J_1J_2J_3]$$

Expressions for the product terms in Equations 3.5.2 and 3.5.3 can be obtained by multiplying the right-hand sides of Equations 3.5.1 and using idempotent cancellation. Thus,

$$J_1J_2 \overset{st}{=} [U_{100}U_{21\vec{1}}\ U_{212}U_{221}U_{222}] \times$$

$$[V_{11\vec{3}}\ V_{114}V_{21\vec{3}}\ V_{214}V_{12\vec{3}}\ V_{124}V_{22\vec{3}}\ V_{224}]$$

$$J_1 J_3 \overset{st}{=} [U_{100} U_{21\vec{1}}\, U_{212} U_{221} U_{222} U_{231} U_{232}] \times$$

$$[V_{11\vec{3}}\, V_{114} V_{21\vec{3}}\, V_{214} V_{13\vec{3}}\, V_{134} V_{23\vec{3}}\, V_{234}]$$

(3.5.4)
$$J_2 J_3 \overset{st}{=} [U_{100} U_{21\vec{1}}\, U_{212} U_{221} U_{222} U_{231} U_{232}] \times$$

$$[V_{12\vec{3}}\, V_{124} V_{22\vec{3}}\, V_{224} V_{13\vec{3}}\, V_{134} V_{23\vec{3}}\, V_{234}]$$

$$J_1 J_2 J_3 \overset{st}{=} [U_{100} U_{21\vec{1}}\, U_{212} U_{221} U_{222} U_{231} U_{232} V_{11\vec{3}}\, V_{114}] \times$$

$$[V_{21\vec{3}}\, V_{214} V_{12\vec{3}}\, V_{124} V_{22\vec{3}}\, V_{224} V_{13\vec{3}}\, V_{134} V_{23\vec{3}}\, V_{234}]$$

Let the availability of component C_k at time t_{00} be α_{k00}, its conditional reliability for phase F_{ij}, $i=0$, $j=1,\ldots,a$; $i=1,\ldots,r$, $j=1,\ldots,b$, be π_{kij}, and its unconditional reliability through the end of the same phase be Π_{kij}, where

(3.5.5)
$$\begin{aligned} \Pi_{kij} &= P[X_{kij}=1] = P[X_{kij}=1 \mid X_{ki,j-1}=1] \cdots P[X_{k00}=1] \\ &= \pi_{kij}\pi_{ki,j-1} \cdots \pi_{k01}\alpha_{k00} \end{aligned}$$

Similarly, let the availability of component $D_k^{(i)}$, $i=1,\ldots,r$, at time t_{i0} be β_{ki0}, its conditional reliability for phase F_{ij}, $j=1,\ldots,c$, be ω_{kij}, and its unconditional reliability through the end of that phase be Ω_{kij}, where

(3.5.6)
$$\begin{aligned} \Omega_{kij} &= P[Y_{kij}=1] = P[Y_{kij}=1 \mid Y_{ki,j-1}=1] \cdots P[Y_{ki0}=1] \\ &= \omega_{kij}\omega_{ki,j-1} \cdots \omega_{ki1}\beta_{ki0} \end{aligned}$$

Then taking expected values in Equations 3.5.1 and 3.5.4 yields

$$E(J_1) = \alpha_{100}\Pi_{212}\Omega_{114}\Omega_{214}$$

$$E(J_2) = \alpha_{100}\Pi_{222}\Omega_{114}\Omega_{214}\Omega_{124}\Omega_{224}$$

$$E(J_3) = \alpha_{100}\Pi_{232}\Omega_{134}\Omega_{234}$$

(3.5.7) $$E(J_1J_2) = \alpha_{100}\Pi_{222}\Omega_{114}\Omega_{214}\Omega_{124}\Omega_{224}$$

$$E(J_1J_3) = \alpha_{100}\Pi_{232}\Omega_{114}\Omega_{214}\Omega_{134}\Omega_{234}$$

$$E(J_2J_3) = \alpha_{100}\Pi_{232}\Omega_{124}\Omega_{224}\Omega_{134}\Omega_{234}$$

$$E(J_1J_2J_3) = \alpha_{100}\Pi_{232}\Omega_{114}\Omega_{214}\Omega_{124}\Omega_{224}\Omega_{134}\Omega_{234}$$

If the component availabilities and conditional phase reliabilities are as given in Figure 3.2, then the unconditional reliabilities shown in Figure 3.3 can be calculated using Equations 3.5.5 and 3.5.6. Then, from Equations 3.5.7, $E(J_1) = .403$, $E(J_2) = .344$, $E(J_3) = .295$, $E(J_1J_2) = .184$, $E(J_1J_3) = .157$, $E(J_2J_3) = .157$, and $E(J_1J_2J_3) = .084$. Substitution into Equations 3.5.3 yields the distribution

$$P[N = 0] = .372$$

$$P[N = 1] = .298$$

$$P[N = 2] = .246$$

$$P[N = 3] = .084$$

Finally, from Equation 3.2.4, the expected number of objectives accomplished during the mission is $E[N] = 1.042$. □

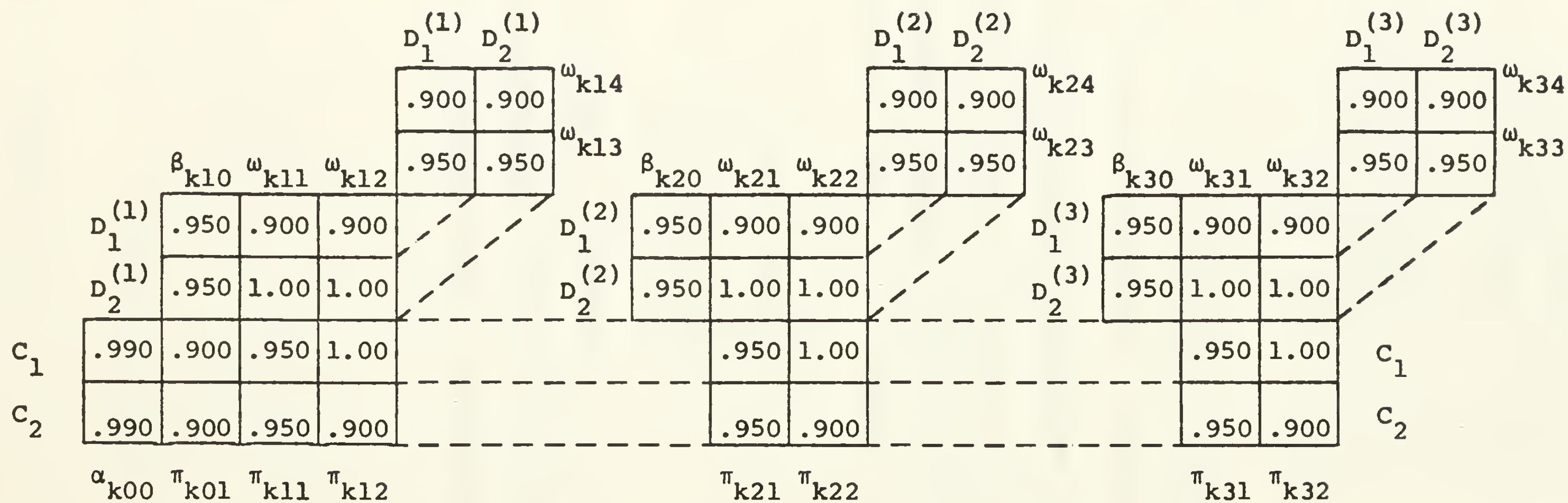


Figure 3.2. Component availabilities and conditional phase reliabilities for Example 3.2.

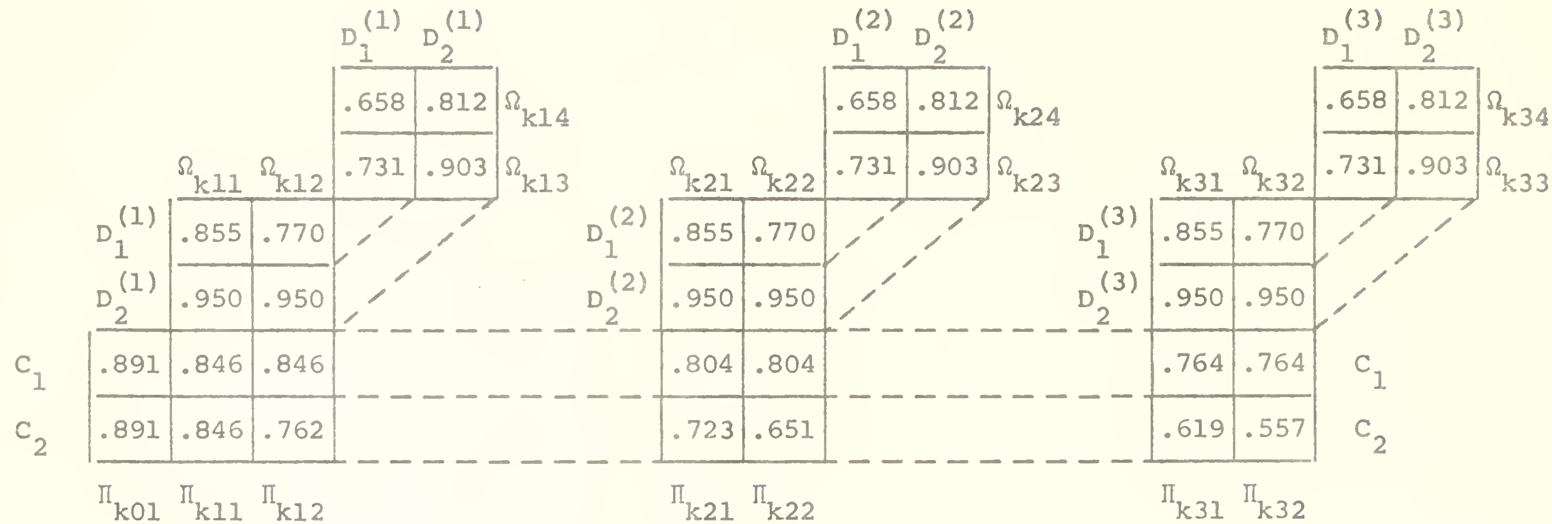


Figure 3.3. Component unconditional phase reliabilities for Example 3.2.

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